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CONCEPT OF THERMAL ENERGY FLOW IN THREE- DIMENSIONAL SOLIDS

Ibitola G. A.[1], Ajanaku Olanrewaju[2] and Ayegbusi Oluwole[3]

[1, 2]Department of Physical Sciences,
[3]Department of Mathematical Sciences,
Ondo State University of Science and Technology,
Okitipupa, Nigeria.

[1] ibitolaieee@gmail.com, [2] lanrayus@yahoo.com, [3]woleayegbusi@gmail.com

Abstract: It has been observed that, hitherto only 1-dimensional and 2-dimensional heat flows in solids are treated, hence in this paper an attempt is made to present the analysis of heat flow in 3-dimensional solids. In this paper, we considered and solved the problems of heat flow in 1-dimensional and 2-dimensional solids. We are interested here in temperature distribution functions, while a lot of literature has been written on heat energy distribution function. Further we solved the problem of heat flow theoretically and computationally in a three-dimensional solid bar, we then identified the parameters upon which the heat flow depends with a view of achieving its controllability. The importance of this research work lies in its provision of some utility data when designing heat pipes for homes and industries. We had used the method of separation of variables for both 1-dimensional and 3-dimensional heat conductions, while we had used the Laplace transform of equations for the 2- dimensional heat flow.

Index Terms: Thermal Energy, energy flow, 3D Solids, energy flow in solids.

I. INTRODUCTION

For the case of 1-dimensional heat flow, let the heat flow occur along a solid bar of uniform cross-section, in the direction perpendicular to the cross-section. Let us take one end of the solid bar as origin and the direction of heat flow is along the x-axis of the Cartesian system of coordinates. Let the temperature of the solid bar at any time t at a point x distance from the origin be $\phi(x,t)$. Then the equation of 1-dimensional thermal energy flow is

$$\frac{\delta\phi}{\delta t} = \frac{c^2\delta^2}{\delta x^2} \qquad \text{(i)}$$

Equation (i) is only valid if and only if the heat flows along y- and z- axes are negligible, which may not be of any practical use. For the 2-dimensional thermal energy flow case, let us consider the heat flow in a metal plate of uniform thickness, in the directions parallel to the length and breadth of the plate. There is no heat flow along the normal to the plane of the rectangular plate. (Refer to Figures 2 and 3). There are two major differences between electrical energy conduction and thermal energy flow.

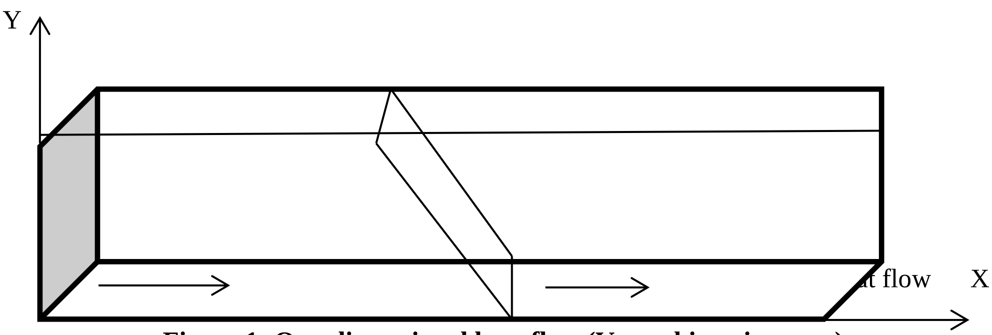

Figure 1: One-dimensional heat flow (Very thin wire case).

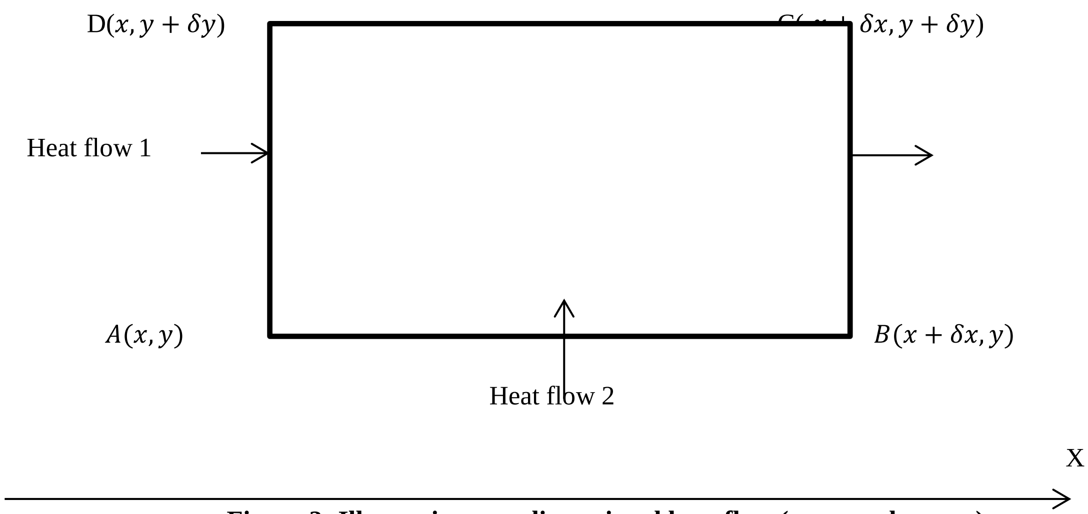

Figure 2: Illustrating two-dimensional heat flow (rectangular case).

In solids, for electrical and wave equation, we have:

$$\frac{\delta^2 y}{\delta t^2} = c^2 \frac{\delta^2 y}{dx^2} \qquad \text{(ii)}$$

Where y = displacement; x = distance from origin along x axis: c = velocity.

Thus from equations (i) and (ii), it can be seen that the velocity function c for electrical /wave conduction in a solid conductor is greater than velocity function c for thermal energy conduction in a solid conductor. Further, the conduction of electrical/wave energy in a solid is guided by the phenomenon of "skin depth or effect" (i.e., the electrical/wave energy is concentrated towards the periphery of the conductor). However, for the conduction of heat energy in a solid, the heat energy is uniformly distributed across the cross section and some temperature gradient is sustained between both ends of the solid conductor, depending upon the composition, nature of the solid and the heat intensity provided.

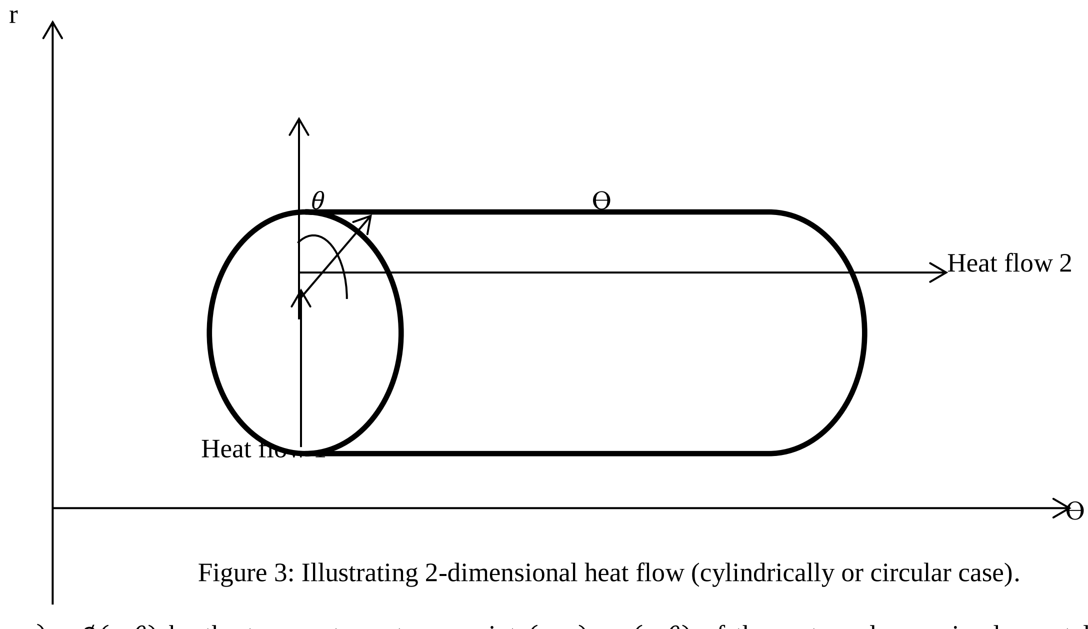

Figure 3: Illustrating 2-dimensional heat flow (cylindrically or circular case).

Let $\emptyset(x, y) or\ \emptyset(r, \theta)$ be the temperature at any point (x, y) or (r, θ) of the rectangular or circular metal plate at any time t which is given by:

$$\frac{\partial \emptyset}{\delta t} = c^2 \left(\frac{\delta^2 \emptyset}{\delta x^2} + \frac{\delta^2 \emptyset}{\delta y^2}\right) \qquad \text{(iii)}$$

Or

$$\frac{\partial \emptyset}{\delta t} = c^2 \left(\frac{\delta^2 \emptyset}{\delta r^2} + \frac{1}{r}\frac{\delta \emptyset}{\delta r} + \frac{1}{r^2}\frac{\delta^2 u}{\delta \theta^2}\right) \qquad \text{(iv)}$$

In the steady or equilibrium state, $\emptyset$ does not vary with time t.

Therefore:

$$\frac{\partial \emptyset}{\delta t} = 0 \qquad \text{(v)}$$

Equations (iii) and (iv) become respectively:

$$\frac{\delta^2 \emptyset}{\delta x^2} + \frac{\delta^2 \emptyset}{\delta y^2} = 0 \qquad \text{(vi)}$$

$$\frac{\delta^2 \emptyset}{\delta r^2} + \frac{1}{r}\frac{\delta \emptyset}{\delta r} + \frac{1}{r^2}\frac{\delta^2 \emptyset}{\delta \theta^2} = 0 \qquad \text{(vii)}$$

Equations (vi) and (vii) are called temperature Laplace equations in two dimensions, (x, y) or (r, θ). These equations would be solved in sections 3 and 4 of this paper. The proposed solution would be presented in section 5 of this paper for the three-dimensional heat flow in Cartesian coordinates wherein the heat flows in all directions are not negligible but significant.

II. ONE-DIMENSIONAL THERMAL ENERGY FLOW

Let us now define this case very clearly. In this case, we have a rod of length L with fully insulated sides, being initially subjected to a uniform temperature $\emptyset$. Its ends are suddenly cooled to 0°C, and then one end is now being allowed to warm up. We shall now attempt to find an expression for its temperature function $\emptyset(x, t)$. Let the equation for the heat conduction in one dimension be:

$$\frac{\delta \emptyset}{\delta t} = c^2 \frac{\delta^2 \emptyset}{\delta x^2} \qquad \text{(viii)}$$

Let $\emptyset = X.T$, where X is a function x alone and T is a function of t alone. Then

$$\frac{\delta \emptyset}{\delta t} = X\frac{dT}{dt} \qquad \text{(ix)}$$

and

$$\frac{\delta^2 \emptyset}{\delta x^2} = T\frac{d^2 X}{dx^2} \qquad \text{(x)}$$

Substituting (ix) and (x) in (viii), we obtain:

$$X\frac{dT}{dt} = c^2 T\frac{d^2 X}{dx^2}$$

Or

$$\frac{1}{c^2 T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2 X}{dx^2} \qquad \text{(xi)}$$

Let each side of equation (xi) be equal to $(-\sigma^2)$. Then we get:

$$\frac{1}{c^2 T}\frac{dT}{dt} = -\sigma^2$$

Or $\frac{dT}{dt} + \sigma^2 c^2 T = 0$ (xii)

and

$$\frac{1}{x}\frac{d^2 X}{dx^2} = -\sigma^2$$

Or $\frac{d^2X}{dx^2} + -\sigma^2 X = 0$ (xiii)

On solving (xii) and (xiii), we have:

$$T = c_1 e^{-\sigma^2 c^2 t}$$

and

$$X = c_2 \cos \quad \sigma x + c_3 \sin \quad \sigma x$$

$$\emptyset = c_1 e^{-\sigma^2 c^2 t}(c_2 cos\sigma x + c_3 sin\sigma x) \qquad \text{(xiv)}$$

Putting $x = 0,\ \emptyset = 0$ in equation (xiv), we obtain:

$$0 = c_1 e^{-\sigma^2 c^2 t}.(c_2)$$

$$\Longrightarrow c_2 = 0\ (since\ c_1 \neq 0)$$

Thus equation (xiv) becomes:

$$\emptyset = c_1 e^{-\sigma^2 c^2 t}.c_3 sin\sigma x \qquad \text{(xv)}$$

On putting $x = l, \emptyset = 0$ in equation (xv), we get:

$$0 = c_1 e^{-\sigma^2 c^2 t}.c_3 sin\sigma l$$

$\Rightarrow \sin \sigma l = 0 = sinn\pi$

$$\Rightarrow \sigma l = n\pi;$$

$$\sigma = \frac{n\pi}{l}\ (\,n\ is\ an\ interger)$$

Hence, equation (xv) becomes:

$$\emptyset = c_1 c_3 e^{-\left(\frac{n^2c^2\pi^2 t}{l^2}\right)} \sin {}^{n\pi x}/_{l}$$

$$= b_n e^{-(n^2c^2\pi^2 t/l^2)} \sin {}^{n\pi x}/_{l} \qquad \text{(xvi)}$$

Where $b_n = c_1 c_3$.

Equation (xvi) satisfies the given condition for all integral values of n. Hence, the temperature function along the rod is:

$$\emptyset(x,t) = \textstyle\sum_{n=1}^{\infty} b_n e^{-(n^2c^2\pi^2 t/l^2)} \sin {}^{n\pi x}/_{l}$$

$$= \emptyset_o e^{-(n^2c^2\pi^2 t/l^2)} \qquad \text{(xvii)}$$

Where $\emptyset = \emptyset_o\ when\ t = 0$

III. TWO-DIMENSIONAL THERMAL ENERGY FLOW

It should be noted that $\emptyset(x,t)$ defines the temperature distribution function along the solid bar, while ${}^{dQ}/_{dt}$ defines the quantity of heat energy flow per second along the bar. ${}^{dQ}/_{dt}$ is given by:

$${}^{dQ}/_{dt} = KA\,{}^{dQ}/_{dx} \qquad \text{(xviii)}$$

Where: K = thermal conductivity of the solid conductor; A = cross sectional area of the bar or rod; ${}^{dQ}/_{dx}$ =temperature gradient along the bar or rod.

3.1 TWO-DIMENSIONAL THERMAL ENERGY FLOW USING CARTESIAN COORDINATES

In this case, we have the following conditions:

1. The solid conductor (e.g., a metal plate) is completely lagged.
2. There is no flow of heat energy in the z- axis direction.
3. The thermal system is in a state of equilibrium.

   Thus, we are now required to solve the thermal energy flow equation:

$$\frac{\delta^2\emptyset}{\delta x^2} + \frac{\delta^2\emptyset}{\delta y^2} = 0 \qquad \text{(xix)}$$

Which satisfies the conditions

$\emptyset(0, y) = 0$; $\emptyset(x, a) = sin\left(\frac{n\pi x}{l}\right)$.

$\emptyset(l, y) = 0$;

$\emptyset(x, 0) = 0$.

The solution is presented as follows:

The two dimensional thermal energy flow equation is

$$\frac{\delta^2\emptyset}{\delta x^2} + \frac{\delta^2\emptyset}{\delta y^2} = 0 \qquad \text{(xx)}$$

Let $\emptyset = X(x).Y(y)$ (xxi)

Then equation (xx) becomes

$$X^{II}Y + XY^{II} = 0$$

Or $\frac{X^{II}}{X} = \frac{-Y^{II}}{Y} = -\sigma^2$

Therefore:

$$X^{II} = -\sigma^2 X$$

$$\Longrightarrow X^{II} + \sigma^2 X = 0 \qquad \text{(xxii)}$$

Similarly, we have:

$$Y^{II} = \sigma^2 Y$$

$$\Longrightarrow Y^{II} - \sigma^2 Y = 0 \qquad \text{(xxiii)}$$

The auxiliary equation (AE) of (xxii) is:

$$m^2 + \emptyset^2 = 0$$

$$\Longrightarrow m = \pm i\sigma$$

Therefore:

$$X = c_1 cos\sigma x + c_2 sin\sigma x \qquad \text{(xxiv)}$$

The AE of equation (xxiv) is:

$m^2 - \sigma^2 = 0$

Or $m = \pm\sigma$

Hence, we have:

$$Y = c_3 e^{\sigma y} + c_4 e^{-\sigma y} \qquad \text{(xxv)}$$

Substituting the values of X and Y into equation (xxi), we get:

$$\emptyset = (c_1 cos\sigma x + c_2 sin\sigma x).(c_3 e^{\sigma y} + c_4 e^{-\sigma y}) \qquad \text{(xxvi)}$$

Putting x = 0, $\emptyset = 0$ in equation (xxv), we get:

$0 = c_1(c_3 e^{\sigma y} + c_4 e^{-\sigma y})$

$\Longrightarrow c_1 = 0.$

Thus equation (xxvi) reduces to:

$$\emptyset = c_2 sin\sigma x(c_3 e^{\sigma y} + c_4 e^{-\sigma y}) \qquad \text{(xxvii)}$$

Substituting $x = l$, $\emptyset = 0$, we get:

$0 = c_2 sin\sigma l.(c_3 e^{\sigma y} + c_4 e^{-\sigma y})$

Since $c_2 \neq 0$, therefore:

$sin\sigma l = sinn\pi = 0$

$\Longrightarrow \sigma l = n\pi$

$\sigma = \frac{n\pi}{l}$

Equation (xxvii) now becomes:

$$\emptyset = c_2 sin\left({}^{n\pi x}/_{l}\right).\left(c_3 e^{{}^{n\pi y}/_{l}} + c_4 e^{-{}^{n\pi y}/_{l}}\right) \qquad \text{(xxviii)}$$

Substituting u =0 and y = 0 in equation (xxviii), we have:

$0 = c_2 sin\left({}^{n\pi x}/_{l}\right).(c_3 + c_4)$

$c_3 + c_4 = 0 \Longrightarrow c_3 = -c_4.$

Equation (xxviii) now becomes:

$$\emptyset = c_2 c_3 sin\left({}^{n\pi x}/_{l}\right).\left\{e^{{}^{n\pi y}/_{l}} - e^{-{}^{n\pi y}/_{l}}\right\} \qquad \text{(xxix)}$$

Substituting y = a, and u(x, y) = $sin\left({}^{n\pi x}/_{l}\right)$ in equation (xxix), we get:

$sin\left({}^{n\pi x}/_{l}\right) = c_2 c_3 sin\left({}^{n\pi x}/_{l}\right)\left\{e^{({}^{n\pi a}/_{l})} - e^{-({}^{n\pi a}/_{l})}\right\}$

$\Longrightarrow c_2 c_3 = \frac{1}{\left\{e^{({}^{n\pi a}/_{l})} - e^{-({}^{n\pi a}/_{l})}\right\}}$ (xxx) Substituting this value for $c_2 c_3$ in equation (xxix), we arrive at:

$$\emptyset(x,y) = sin\left({n\pi x}/{l}\right)\frac{\left[e^{\left({n\pi y}/{l}\right)} - e^{-\left({n\pi y}/{l}\right)}\right]}{\left[e^{\left({n\pi a}/{l}\right)} - e^{-\left({n\pi a}/{l}\right)}\right]}$$

$$= sin\left({n\pi x}/{l}\right).\frac{sinh\left({n\pi y}/{l}\right)}{sinh\left({n\pi a}/{l}\right)} \qquad \text{(xxxi)}$$

3.2 TWO-DIMENSIONAL HEAT FLOW USING POLAR COORDINATES

In this case, the solid conductor is a solid or hollow metallic plate of uniform circular cross-section. The sides of the solid conductor are insulated, while the ends bear the source and the sink. The solid conductor is then subjected to the following conditions:

$\emptyset(r,0) = 0;$ $\qquad 0 \le r \le a$

$\emptyset(r,+\pi) = 0;$ $\qquad 0 \le r \le a$

$\emptyset(r,-\pi) = 0;$ $\qquad 0 \le r \le -a$

$\emptyset(a,\theta) = C,\ a\ constant.$

Let the centre of origin O be the centre of the cross sectional circle of the cylinder at the sending end. Let $\emptyset(r,\ \theta)$ be the steady state temperature at any point $p\ (r,\ \theta)$ and $\emptyset$ satisfies the equation:

$$r^2\frac{\delta^2\emptyset}{\delta r^2} + r\frac{\delta\emptyset}{\delta r} + \frac{\delta^2\emptyset}{\delta\theta^2} = 0 \qquad \text{(xxxii)}$$

Let $\emptyset = R(r).T(\theta)$. (xxxiii)

Then we have:

$$\frac{\delta\emptyset}{\delta r} = \frac{dR}{dr}.T(\theta)$$

$$\frac{\delta^2\emptyset}{\delta r^2} = \frac{d^2R}{dr^2}.T(\theta)$$

$$\frac{\delta\emptyset}{\delta\theta} = R(r).\frac{dT}{d\theta}$$

$$\frac{\delta^2\emptyset}{\delta\theta^2} = R(r).\frac{d^2T}{d\theta^2}$$

Substituting the values of $\frac{\delta\emptyset}{\delta r}$, $\frac{\delta^2\emptyset}{\delta r^2}$, and $\frac{\delta^2\emptyset}{\delta\theta^2}$ for equation (xxxii), we have:

$$r^2\frac{d^2R}{dr^2}.T(\theta) + r\frac{dR}{dr}.T(\theta) + R(r).\frac{d^2T}{d\theta^2} = 0$$

$$\left[r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr}\right].T + R\frac{d^2T}{d\theta^2} = 0$$

$$\frac{\left[r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr}\right]}{R} = \frac{-1}{T}.\frac{d^2T}{d\theta^2} = h \qquad \text{(xxxiv)}$$

For R, we have:

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - hR = 0$$

$Let\ r = e^z$. Then

$D^2R + DR - hR = 0$

$(D^2 + D - h)R = 0$

$\Longrightarrow \{(D)[D + 1]\} = \pm h$

$\Longrightarrow D = \pm\sqrt{h}$

Hence we get:

$R = c_1 e^{\sqrt{h}z} + c_1 e^{-\sqrt{h}z}$

Or $R = c_1 r^{\sqrt{h}} + c_2 r^{-\sqrt{h}}$ (xxxv)

To obtain the value of T, we proceed as follows:

$\frac{d^2T}{d\theta^2} + hT = 0$ (xxxvi)

$(D^2 + h)T = 0$

$D^2 + h = 0$

$\Longrightarrow D = \pm i\sqrt{h}$

Hence, we get:

$T = c_3 \cos(\sqrt{h},\ \theta) + c_4 \sin(\sqrt{h},\ \theta)$ (xxxvii)

Thus, we finally obtained:

$\emptyset(r,\theta) = \left(c_1 r^{\sqrt{h}} + c_2 r^{-\sqrt{h}}\right)\left[c_3 \cos(\sqrt{h},\ \theta) + c_4 \sin(\sqrt{h},\ \theta)\right]$ (xxxviii)

IV. THE PROPOSED THREE-DIMENSIONAL HEAT FLOW

In this section, we shall solve the problem, in Cartesian coordinates, for three-dimensional heat solid conductor which is perfectly lagged or insulated.

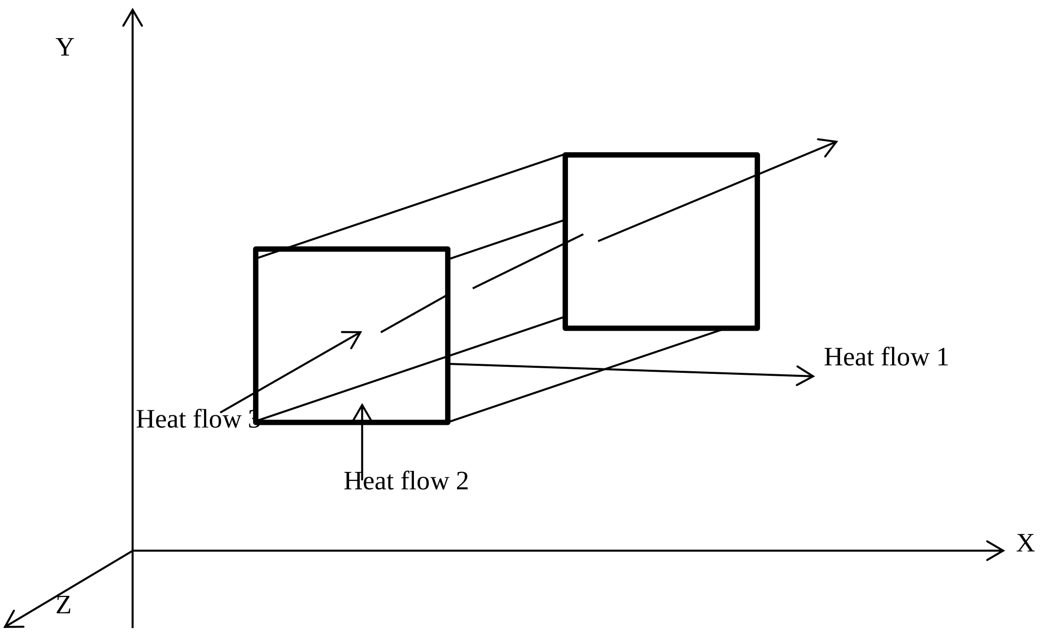

Figure 4: Heat flow in a rectangular bar.

Referring to figure 4, consider the heat flow in a metallic bar of uniform rectangular cross-section or thickness, in the directions parallel to height, breadth and length of the bar. Let $\emptyset\ (x, y, z)$ be the temperature at any point (x, y, z) in the bar. At time t, $\emptyset$ satisfies the equation:

$$\frac{\delta\emptyset}{\delta t} = c^2\left(\frac{\delta^2\emptyset}{\delta x^2} + \frac{\delta^2\emptyset}{\delta y^2} + \frac{\delta^2\emptyset}{\delta z^2}\right) \qquad \text{(xxxix)}$$

In the steady state, $\emptyset$ does not fluctuate with time t.

Therefore:

$$\frac{\delta\theta}{\delta t} = 0$$

$$\Rightarrow c^2\left(\frac{\delta^2\emptyset}{\delta x^2} + \frac{\delta^2\emptyset}{\delta y^2} + \frac{\delta^2\emptyset}{\delta z^2}\right) = 0$$

Or $$\frac{\delta^2\emptyset}{\delta x^2} + \frac{\delta^2\emptyset}{\delta y^2} + \frac{\delta^2\emptyset}{\delta z^2} = 0 \qquad \text{(xL)}$$

Which satisfies the conditions:

$\emptyset\ (0, y, z) = 0;$

$\emptyset\ (x, 0, z) = 0;$

$\emptyset\ (x, y, 0) = 0;$

$\emptyset\ (l, y, z) = 0;$

$\emptyset\ (x, l_2, z) = 0;$

$\emptyset\ (x, y, l_3) = \frac{sinn\pi x}{l_1}\frac{sinn\pi y}{l_2}.$

Let $\emptyset = X(x).Y(y).Z(z)$ (xLi)

Substituting the values of $\frac{\delta^2\emptyset}{\delta x^2}, \frac{\delta^2\emptyset}{\delta y^2}, \frac{\delta^2\emptyset}{\delta z^2}$ into equation (xL), we get:

$$YZ\frac{\delta^2 X}{\delta x^2} + XZ\frac{\delta^2 Y}{\delta y^2} + XY\frac{\delta^2 Z}{\delta z^2} = 0 \qquad \text{(xLii)}$$

Dividing through equation (xLii) by XYZ, we have:

$$\frac{YZ}{XYZ}X^{II} + \frac{XZ}{XYZ}Y^{II} + \frac{XY}{XYZ}Z^{II} = 0$$

$$\frac{X^{II}}{X} + \frac{Y^{II}}{Y} + \frac{Z^{II}}{Z} = 0 \qquad \text{(xLiii)}$$

$$\frac{X^{II}}{X} = \frac{-Y^{II}}{Y} = -\sigma_1^2 \qquad \text{(xLiv-a)}$$

$$\frac{X^{II}}{X} = \frac{-Z^{II}}{Z} = -\sigma_2^2 \qquad \text{(xLiv-b)}$$

$$X^{II} = -\sigma_1^2 X$$

$$\Rightarrow X^{II} + \sigma_1^2 X = 0 \qquad \text{(xLv – a)}$$

$$X^{II} = -\sigma_2^2$$

$$\Rightarrow X^{II} + \sigma_2^2 = 0 \qquad \text{(xLv – b)}$$

Auxiliary equations of (xLv a –b) are:

$m_1^2 + \sigma_1^2 = 0$

And $m_2^2 + \sigma_2^2 = 0$

$m_{1x} = \pm i\sigma 1$, and $m_{2x} = \pm i\sigma_2$,

Thus, we get:

$X = c_1 cos\sigma_1 x + c_2 sin\sigma_1 x$ (XLVI–a)

And $X = c_3 cos\sigma_2 x + c_4 sin\sigma_4 x$ (XLVI –b)

$Y^{II} = \sigma_1^2 Y$ or $Y^{II} - \sigma_1^2 Y = 0$

$Z^{II} = \sigma_1^2 Z$ or $Z^{II} - \sigma_1^2 Z = 0$

The auxiliary equations for the above equations are:

$m_y - \sigma_1^2 = 0 \Longrightarrow m_y = \pm\sigma_1$

$m_z - \sigma_2^2 = 0 \Longrightarrow m_y = \pm\sigma_2$

Hence, we have:

$Y = c_5 e^{\sigma_1 y} + c_6 e^{-\sigma_1 y}$ (XLVII)

and

$Z = c_7 e^{\sigma_2 z} + c_8 e^{-\sigma_2 z}$ (XLVIII)

Thus, $\emptyset$ is given by (in three-dimensions):

$\emptyset\,(x, y, z) = XYZ = [c_1 cos\sigma_1 x + c_2 sin\sigma_1 x].[c_3 cos\sigma_2 x + c_4 sin\sigma_4 x].[c_5 e^{\sigma_1 y} + c_6 e^{-\sigma_1 y}][c_5 e^{\sigma_1 y} + c_6 e^{-\sigma_1 y}]$

(XLIX)

Where $c_1, c_2, c_3, \ldots c_7, c_8$ are constants, which depend on the nature of the solid conductor. A C++ program can be designed to compute and display the value of $\emptyset\,(x, y, z)$ using files.

V. CONCLUDING REMARKS

In conclusion, we have presented as commonly and normally treated in texts, both one-dimensional and two-dimensional heat flows in solid conductors. We then took a step further to present using the method of Laplace equation and separation of variables to obtain a useful expression for temperature distribution function, $\emptyset\,(x, y, z)$, in a solid conductor in three dimensional. A C++ program can be designed using files and mathematical functions to compute and display the values of $\emptyset\,(x, y, z)$ *at designated points along the solid* pipe or conductor for any given case. Thus, the heat flows depend on temperature gradients (σ_1 *and* σ_2) and on the solid conductor constants($c_1, c_2, c_3, \ldots . c_8$).

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