CERTAIN QUADRUPLE SERIES EQUATIONS

Kuldeep Narain

School of Quantitative Sciences
College of Arts and Sciences
Universiti Utara Malaysia
06010 UUM, Sintok, Kedah Darul Aman, Malaysia.

***Abstract*- In this paper, we have obtained the solution of the quadruple series equations involving Jacobi polynomials as kernel. These equations are finally reduced to Fredholm integral equations of second kind.**

***Index Terms*— Integral Equation, Series Equation, Jacobi polynomial**

I. INTRODUCTION

In this paper, the solution of the following quadruple series equations have been derived

$$\sum_{n=o}^{\infty} \frac{A_n\,(1+H_n)P_n^{(\alpha,\beta)}(x)}{(\beta+1)_n\,\Gamma(\alpha-\mu+n+1)} = f_1(x), \qquad -1 < x < a \tag{1.1}$$

$$\sum_{n=o}^{\infty} \frac{A_n\,P_n^{(\lambda,\delta)}(x)}{(\lambda+1)_n\,\Gamma(\beta-\mu+n+1)} = f_2(x), \qquad a < x < b \tag{1.2}$$

$$\sum_{n=o}^{\infty} \frac{A_n\,(1+H_n)P_n^{(\alpha,\beta)}(x)}{(\beta+1)_n\,\Gamma(\alpha-\mu+n+1)} = f_3(x), \qquad b < x < c \tag{1.3}$$

$$\sum_{n=o}^{\infty} \frac{A_n\,P_n^{(\lambda,\delta)}(x)}{(\lambda+1)_n\,\Gamma(\beta-\mu+n+1)} = f_4(x), \qquad c < x < 1 \tag{1.4}$$

Where $f_1(x)$ to $f_4(x)$ are prescribed functions, sequences $\{A_n\}$ is to be determined and the parameters α, β, λ, δ, μ satisfy the conditions $\delta - \mu - \beta > o$, $\alpha > -1$, $\beta > -1$, $\lambda > -1$ and $\delta > -1$.

II. SOME USEFUL RESULTS

Here we listed some results for ready reference:

(i) The orthogonality relation for Jacobi polynomials is

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x).\,P_m^{(\alpha,\beta)}(x)\,(1-x)^{\alpha}\,(1+x)^{\beta} = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)\delta_{m,n}}{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)} \tag{2.1}$$

where, $\delta_{m,n}$ is kronecker's delta and $\alpha > -1$, $\beta > -1$.

(ii) The series

$$S(r,x) = \sum_{n=o}^{\infty} \frac{(\lambda+1)_n\,\Gamma(\beta+\mu+n+1)(2n+\alpha+\beta+1)\Gamma(n+1)}{2^{\alpha+\beta+1}\Gamma(n+\delta+1)\,(n+\lambda+1)(\beta+1)_n}$$

$$\frac{\Gamma(n+\alpha+\beta+1)}{\Gamma(\alpha-\mu+n+1)}\, P_n^{(\alpha,\beta)}(x).P_m^{(\lambda,\delta)}(r), \qquad (2.2)$$

$$= a_n^*\,(1-r)^{-\delta}\,(1+x)^{\beta}\int_{-1}^{w} n(y)(r-y)^{\mu-1}.(x-y)^{\beta+\mu-\delta-1}dy$$

$$= a_n^*\,(1-r)^{-\delta}\,(1+x)^{-\beta}\,S_w^{-1}(r,x), \qquad (2.3)$$

Where $w = \min\ (x, y)$, $n(y) = (1+y)^{-\delta-\mu}\,(1-y)^{-\lambda-\mu}$

and

$$a_n^* = \frac{\Gamma(1+\beta)\Gamma(\beta-\delta+\mu)\Gamma(\beta+\mu+n+1)[(n+\alpha+1)]^2}{\Gamma(\mu)\,\Gamma(\lambda+1)\,\Gamma(n+\alpha-\mu+1)\Gamma(n+\delta-\mu+1)\Gamma(\lambda+\mu+n+1)}$$

It being assumed that the parameters are so constrained that a_n^* Is independent of

(iii) It f(x) and $f^1(x)$ are continuous in $a \le x \le b$ and if $o < \sigma < 1$ then the solution of Able integral equations

$$f(x) = \int_a^x \frac{F(y)}{(x-y)^{\sigma}}\,dy, \qquad (2.4)$$

and

$$f^1(x) = \int_x^b \frac{F(y)}{(y-x)^{\sigma}}\,dy, \qquad (2.5)$$

are given by

$$F(y) = \frac{\mathrm{Sin}\,(\sigma\pi)}{\lambda}\frac{d}{dy}\int_a^y \frac{f(x)dx}{(y-x)^{1-\sigma}}, \qquad (2.6)$$

and

$$F(y) = \frac{\mathrm{Sin}\,(\sigma\pi)}{\pi}\frac{d}{dy}\int_y^b \frac{f(x)dx}{(y-x)^{1-\sigma}}, \qquad (2.7)$$

respectively

III. SOLUTION OF EQUATIONS

Let us suppose

$$\sum_{n=o}^{\infty}\frac{A_n\,P_n^{(\lambda,\delta)}(x)}{(\lambda+1)_n\,\Gamma(\beta+\mu+n+1)} = \begin{cases} h(x), -1<x<a, \\ g(x),\ b<x<c, \end{cases} \qquad (3.1)$$

Making use of the orthogonality relation (2.1), we get from (1.2), (1.4) and (3.1),

$$A_n = \frac{(n+1)\,\Gamma(n+\alpha+\beta+1)}{2^{\alpha+\beta+1}\,\Gamma(n+\lambda+1)\Gamma(n+\delta+1)}\Bigg[\int_{-1}^{a}(1-x)^{\lambda}(1+x)^{\delta}h(x)P_n^{(\lambda,\delta)}(x)dx + \int_a^b (1-x)^{\lambda}(1+x)^{\delta}f_2(x)P_n^{(\lambda,\delta)}(x)dx$$
$$+\int_b^c (1-x)^{\lambda}(1+x)^{\delta}g(x)\,P_n^{(\lambda,\delta)}(x)dx$$
$$+\int_c^1 (1-x)^{\lambda}(1+x)^{\delta}f_4(x)P_n^{(\lambda,\delta)}(x)dx\Bigg], \qquad (3.2)$$

Substituting this expression for A_n from (3.2) in (1.1) and (1.3) and interchanging the order of integration and summation, we obtain the equations

$$\int_{-1}^{a}(1-r)^{\lambda}(1+x)^{\delta}\ h(r)[S(r,x)+T(r,x)]dr +$$

$$\int_b^c (1-r)^{\lambda}(1+r)^{\delta}\ g(r)[\,S\,(r,x)+T(r,x)]dr = M(x), -1 \le x \le a, \qquad (3.3)$$

$$\int_{-1}^{a} (1-r)^{\lambda}(1+x)^{\delta}\ h\,(r)[\,S\,(r,x)+T\,(r,x)]\ dr +$$

$$\int_b^c (1-r)^{\lambda}(1+r)^{\delta}\ g(r)[\,S(r,x)+T(r,x)]dr = N(x), b < x < c \qquad (3.4)$$

Where,

$$M(x) = f_1\,(x) - \int_a^b (1-r)^{\lambda}(1+r)^{\delta}\ F_2\,(r)[\,S\,(r,x)+T(r,x)]dr$$

$$-\int_o^a (1-r)^{\lambda}(1+r)^{\delta}\ F_4\,(r)[\,S\,(r,x)+T(r,x)]\ dr, \qquad (3.5)$$

$$N(x) = f_3\,(x) - \int_a^b (1-r)^{\lambda}(1+r)^{\delta}\ F_2\,(r)[\,S\,(r,x)+T(r,x)]dr \qquad (3.6)$$

S (r, x) is same as detained by (2.2) and

$$T(r,x) = \sum_{n=o}^{\infty} \frac{(\lambda+1)_n \Gamma\,(\beta+\mu+n+1)(2n+\alpha+\beta+1)(n+1)}{2^{\alpha+\beta+1}\Gamma\,(n+\delta+1)\,(n+\lambda+1)(\beta+1)_n}$$

$$\frac{\Gamma(n+\alpha+\beta+1)}{\Gamma(\alpha\ -\mu+n+1)}\ H_n\,P_n^{(\alpha,\beta)}(x).\,P_m^{(\lambda,\delta)}(r) \qquad (3.7)$$

It is assumed that the form of H_n is such that T (r,x) converges.
Equation (3.3) and (3.4) can be written as with the help of (2.3)

$$\int_{-1}^{x} \frac{n\,(y)dy}{(x-y)^{1-\beta-\mu+\delta}} \int_y^{a} \frac{(1-r)^{\lambda}\,h\,(r)dr}{(r-y)^{1-\mu}} + \int_b^{c} (1-r)^{\lambda}\ g(r)dr$$

$$\int_{-1}^{x} \frac{n(y)dy}{(x-y)^{1-\beta-\mu+\delta}\ (r-y)^{1-\mu}} + \left\{\int_{-1}^{a} (1-r)^{\lambda}\,(1+x)^{\delta}\,h(r) + \int_b^{c} (1-r)^{\lambda}\,(1+r)^{\delta}\,g\,(r)\right\} T(r,x)dr$$

$$= \frac{(1+x)^{\beta}M(x)}{a_n^*} \qquad (3.8)$$

With the help of (2.6), we find from (3.8)

$$\int_y^{a} \frac{(1-r)^{\lambda}\,h(r)dr}{(r-y)^{1-\mu}} = \frac{M_1\,(y)}{n(y)} - \int_y^{c} \frac{(1-r)^{\lambda}\,g(r)dr}{(r-y)^{1-\mu}} \qquad (3.9)$$

Where,

$$M_1\,(y) = \frac{\mathrm{Sin}\,(1-\beta-\mu+\delta)\pi}{\pi a_n^*}\ \frac{d}{dy}\ \int_{-1}^{y} \frac{(1+x)^{\beta}\,M(x)dy}{(y-x)^{\beta+\mu-\delta}} \qquad (3.10)$$

Using (2.7), we obtain from (3.9)

$$(1-r)^{\lambda}\ h(r) = -\frac{\mathrm{Sin}\ (1-\mu)\pi}{\pi}\frac{d}{dy}\int_{r}^{a}\frac{M_1\,(y)dy}{n(y)(y-r)^{\mu}} +$$

$$\frac{\mathrm{Sin}\ (1-\mu)\pi}{\pi}\frac{d}{dy}\int_{r}^{a}\frac{dy}{(y-r)^{\mu}}\int_{b}^{c}\frac{(1-s)^{\lambda}\ g(s)ds}{(s-y)^{1-\mu}}\ ,-1<r<a \tag{3.11}$$

Now, using the result

$$\frac{d}{dr}\int_{r}^{a}\frac{dy}{(y-r)^{\mu}\ (s-y)^{1-\mu}} = -\frac{(a-r)^{-\mu}}{(s-a)^{-\mu}\,(s-r)}, \tag{3.12}$$

We find that

$$h(r) = M_2(r) - \frac{\mathrm{Sin}\ (1-\mu)\pi}{\pi}\frac{(1-r)^{-\lambda}}{(a-r)^{\mu}}\int_{b}^{c}\frac{(s-a)^{\mu}(1-s)^{\lambda}g(s)}{(s-r)}ds \tag{3.13}$$

Where,

$$M_2\,(r) = -\frac{\mathrm{Sin}\ (1-\mu)\pi}{\pi}\ (1-r)^{-\lambda}\frac{d}{dr}\int_{r}^{a}\frac{M_1\,(y)dy}{n(y)(y-x)^{\mu}}, \tag{3.14}$$

Using (3.4) and (2.3), we find

$$\int_{b}^{x}\frac{n(y)G\,(y)dy}{(x-y)^{1-\beta-\mu+\delta}} + \left\{\int_{-1}^{a}(1-r)^{\lambda}\,(1+x)^{\delta}\ h(r) + \int_{b}^{c}(1-r)^{\lambda}\,(1+r)^{\delta}\,g(r)\right\}T(r,x)dr = \frac{(1+x)^{\mu}N(x)}{a_n^{*}} -$$
$$\int_{-1}^{a}(1-r)^{\lambda}\ h(r)dr\int_{-1}^{r}\frac{n\ (y)dy}{(x-y)^{1-\beta-\mu+\delta}\ (r-y)^{1-\mu}} -$$
$$\int_{-1}^{b}\frac{n\ (y)dy}{(x-y)^{1-\beta-\mu+\lambda}}\int_{b}^{a}\frac{(1-r)^{\lambda}\,g(r)dr}{(r-y)^{1-\mu}}, \qquad b< x<c \tag{3.15}$$

Where,

$$G(y) = \int_{y}^{c}\frac{(1-r)^{\lambda}\ g(r)dr}{(r-y)^{1-\mu}}, \tag{3.16}$$

From (3.16) and (2.7), we find

$$(1-r)^{\lambda}g(r) = -\frac{\mathrm{Sin}\ (1-\mu)\pi}{\pi}\frac{d}{dr}\int_{r}^{c}\frac{G\,(y)dy}{(y-x)^{\mu}}, \tag{3.17}$$

The equation (3.15) has the form (2.4), then using (2.6), it can be written in the form,

$n\ (y)\ G\ (y) = N_1\ (y) + I_1 + I_2,$ $\qquad b < x < c$ (3.18)

Where,

$$N_1\,(y) = \frac{\mathrm{Sin}\ (1-\beta-\mu+\delta)\pi}{\pi\ a_n^{*}}\frac{d}{dy}\int_{b}^{y}\frac{(1+x)^{\beta}\ N(x)dx}{(y-x)^{\beta+\mu-\delta}}, \tag{3.19}$$

$$I_1 = -\frac{\mathrm{Sin}\ (1-\beta-\mu+\delta)\pi}{\pi}\frac{d}{dy}\int_{b}^{y}\frac{dx}{(y-x)^{\beta+\mu-\delta}}$$
$$\int_{-1}^{a}(1-r)^{\lambda}h\ (r)dr.\int_{-1}^{t}\frac{n(t)dt}{(x-t)^{1-\beta-\mu+\delta}\ (r-t)^{1-\mu}}, \tag{3.20}$$

$$I_2 = \frac{\mathrm{Sin}\ (1-\beta-\mu+\delta)\pi}{\pi}\frac{d}{dy}\int_{b}^{y}\frac{dx}{(y-x)^{\beta+\mu-\delta}}$$
$$\int_{-1}^{b}\frac{n(t)dt}{(x-t)^{1-\beta-\mu+\delta}}\int_{b}^{c}\frac{(1-r)^{\lambda}g\ (r)dr}{(r-y)^{1-\mu}}, \tag{3.21}$$

After some manipulation, we get

$$I_1 = -\frac{\text{Sin}\,(I-\beta-\mu+\delta)\pi}{\pi\ (y-b)^{\beta+\mu-\delta}} \int_{-1}^{a} \frac{(b-t)^{\beta+\mu-\delta} n(t)dt}{(y-t)} \int_{y}^{a} \frac{(1-r)^{\lambda} h(r)dr}{(r-t)^{1-\mu}}, \qquad (3.22)$$

Putting the value of the last integral in the above equatin from (3.9), we get

$$I_1 = M_3\,(y) + \frac{\text{Sin}\,(I-\beta-\mu+\delta)\pi}{\pi\ (y-b)^{\beta+\mu-\delta}} \int_{b}^{c} (1-r)^{\lambda}\, g(r)dr$$

$$\int_{-1}^{a} \frac{(b-t)^{\beta+\mu-\delta} n(t)dt}{(y-t)\ (1-r)^{1-\mu}} \qquad (3.23)$$

Where,

$$M_3\,(y) = \frac{\text{Sin}\,(I-\beta-\mu+\delta)\pi}{\pi\ (y-b)^{\beta+\mu-\delta}} \int_{-1}^{a} \frac{M_1\,(t)(b-t)^{\beta+\mu-\delta} dt}{(y-t)} \qquad (3.21)$$

After some manipulaton the equation (3.21) can be written in the form.

$$I_2 = -\frac{\text{Sin}\,(I-\beta-\mu+\delta)\pi}{\pi\ (y-b)^{\beta+\mu-\delta}} \int_{b}^{c} (1-r)^{\lambda}\, g(r)dr \int_{-1}^{b} \frac{(b-t)^{\beta+\mu-\delta} n(t)dt}{(y-t)\ (r-t)^{1-\mu}}, \qquad (3.24)$$

Hence,

$$I_1 + I_2 = M_3\,(y) - \frac{\text{Sin}\,(I-\beta-\mu+\delta)\pi}{\pi\ (y-b)^{\beta+\mu-\delta}} \int_{a}^{b} \frac{n(t)(b-t)^{\beta+\mu-\delta}}{(y-t)} \text{Rdt} \qquad (3.25)$$

Where,

$$R = \frac{\text{Sin}\,(I-\mu)\pi}{\pi}\ (b-t)^{\mu} \int_{b}^{c} \frac{G(r)dr}{(r-b)^{\mu}(r-t)} \qquad (3.26)$$

Using equations (3.23) and (3.24) the equations (3.18) can be written in the form

$$N(y)\ G\ (y) = M_3\,(y) + N_1(y) + \int_{b}^{c} G\ (r)\ k\ (r,y)dr, \qquad b < y < c \qquad (3.27)$$

Where,

$$k(r,y) = \frac{\text{Sin}\,(I-\mu+)\pi \text{Sin}(I-\beta-\mu+\delta)\pi}{\pi^2\ \ (y-b)^{\beta+\mu-\delta}\ (r-b)^{\mu}} \int_{a}^{b} \frac{n(t)(b-t)^{2\mu+\beta-\delta}}{(r-t)(y-t)} \qquad (3.28)$$

Equation (3.27) is a Fredholm integral equation of second kind. From equation (3.27), we can determine G(y), then g(u) and h(u) can be easily obtained from equation (3.18) and equation (3.13) and hence the coefficients A_n can be found.

REFERENCES

[1] Cooke, J.C; Tripal integral equations, Quart. J. Mech. Appl. Math; 16, pp. 193-203 (1963).

[2] Cooke, J.C; The solution of triple and quadruple integral equations and Fourier-Bessel Series, Quart. J. Mech. Appl. Math. 25, pp.247-263 (1972).

[3] Rainville, E.D; Special Functions, Macmillan (1967).

[4] Spence, P.A; On dual series relations involving Laguerre Polynomials, Pacific. J. Maths ; 19, pp.529-533(1966b).