FORMULATION OF FRACTIONAL DIFFERENTIAL EQUATIONS WITH INITIAL CONDITIONS

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Abstract

In this paper, we consider the Riemann-Liouville fractional derivative and the Caputo fractional derivative with particular reference to the initial conditions necessary for the formulation of fractional differential equations. We further determine the Riemann-Liouville and the Caputo fractional derivative of the function $f(x)=1$ *stating their relationship. In conclusion, we use Laplace transform to establish a relationship between the Beta and the gamma functions.*

***Keywords**: Riemann-Liouville, fractional derivative, Caputo, Laplace transform, fractional differential equations and integrals.*

1.0 Introduction

In recent years it has turned out that many phenomena in engineering, Physics, Chemistry and other sciences can be described by models using Mathematical tools from fractional calculus [6,7,8]. In the 19th century Abel formulated an integral equation which led to the development of fractional integro-differential operators by Liouville and Riemann [1, 2, 3, 4].

In the 20th century, fractional derivatives found application in the study of viscoelasticity; diffusion and fluid dynamics problems amongst others with the advent of computers, numerical methods were developed for calculating the approximate solution to FDEs leading to an increase in their applications [5, 6].

The theory of derivatives and integrals of fractional order, some of the most prominent examples are given in a book of Oldham and Spanier [1]. In [9], the studied in details were description and assessment of numerical methods were described by fractional – order derivatives, integrals and differential equations which they provide an algorithm for calculating the Mittag-Leffler function which appears in solution to fractional – order differential equations.

2.0 **Formulation of fractional differential equations**.

By a first order ordinary fractional differential equation we will mean a fractional differential equation whose derivatives are all of order < 1.

A first order fractional differential equation defined in terms of Riemann-Liouville derivatives of f on $[a,b]$ integrate before differentiating $DJ_a^{1-\alpha} f$

And the Caputo fractional derivative on $[a,b]$ is differentiable before integrating $J^{n+1}D^{n+1}f$, hence, for the Riemann-Liouville derivatives will require the fractional integral $\lim_{t\to a} J_a^{1-\alpha} f(t)$ as an initial condition while a first order fractional ordinary differential equation defined in terms of Caputo derivatives will only require $f(a)$ as an initial condition.

$$DJ_a^{1-\alpha} = D\left\{\frac{\Gamma(1)x^{1-\alpha}}{\Gamma(2-\alpha)}\right\} = \frac{x^{-\alpha}}{\Gamma(1-\alpha)} \text{ From } J^{1-\alpha}x^0 = \frac{\Gamma(0+1)x^{1-\alpha+0}}{\Gamma(1-\alpha+0+1)} = \frac{\Gamma(1)x^{1-\alpha}}{\Gamma(2-\alpha)} \text{ hence } J_a^{1-\alpha}D1 = J_a^{1-\alpha}0 = 0$$

(1)

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The two derivatives are related by the formula $D_a^\alpha\left[f-T_{m-1}[f;a]\right]=D_a^\alpha f$ where $T_{m-1}[f;a]$ is the Taylor polynomial of degree m^{-1} centered at a and we shall call $D_a^\alpha f$ the Caputo fractional differential operators of advents, hence, $D_a^\alpha f = J_a^{m-\alpha} D^m f$.

3.0 Riemann-Liouville fractional integral of order $\alpha > 0$, of $f(x)$ over the interval $[0,\infty)$.

The Riemann-Liouville fractional integral of the function f of order $\alpha > 0$, of $f(x)$ over the interval $[0,\infty)$ is given by $J_0^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x f(t)(x-t)^{\alpha-1}dt$ and the condition that f must satisfy on this interval is given by $f\varepsilon L_1[0,\infty)$ which has to be integrated. Generally, we have that $J_a^\alpha f = \int_a^x f(t)(x-t)^{\alpha-1}dt$, $x\varepsilon[a,b]$.

Example: By the use of Laplace transforms show that $J^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}x^{\alpha+\beta}$ where $\beta > 0$.

$J_0^\alpha x^\beta = \frac{1}{\Gamma(\alpha)}\int_0^x t^\beta (x-t)^{\alpha-1}dt$ Using convolution theory,

$$\ell\{J_0^\alpha x^\beta\}=\frac{1}{\Gamma(\alpha)}\ell\{t^\beta\}\ell\{t^{\alpha-1}\}=\frac{1}{\Gamma(\alpha)}\frac{\Gamma(\beta+1)\Gamma(\alpha)}{S^{\beta+1}S^\alpha}=\frac{\Gamma(\beta+1)}{S^{\beta+1+\alpha}}$$

Taking the inverse Laplace transform of both sides, we obtain

$$J_0^\alpha f(x)=\Gamma(\beta+1)\frac{x^{\beta+\alpha}}{\Gamma(\beta+1+\alpha)} \qquad (2)$$

where $\beta > 0$.

4.0 Convergence of Riemann-Liouville fractional integral.

Given that $f_n(x)=\frac{x^2}{1+1/n}+\frac{1}{(2+x)^n}$ show that $\{f_n\}_{n=1}^\infty$

Converges uniformly to

$f(x)=x^2$ on $[0,1]$. Let $\varepsilon > 0$ and consider $|\ f_n(x)-f(x)|=|\sum_{m=n+1}^\infty \frac{x^{m+1/2}}{m!}|\le|\sum_{m=n+1}^\infty \frac{1}{m!}|\le|\sum_{m=n+1}^\infty \frac{1}{(m-1)^2(m-2)!}|$

$$\le\frac{1}{n^2}|\sum_{m=n+1}^\infty\frac{1}{(m-2)!}|\le\frac{1}{n^2}e^1$$

Choose n such that $\frac{e}{n^2}<\varepsilon$, $\left(\frac{e}{\varepsilon}\right)^{1/2}<n$.

Similarly, let $g_n(x)=\frac{x^2}{1+1/n}$.

We shall confirm by fractional integration that $J_0^{1/2}g_n$ converges uniformly to $J_0^{1/2}g$ on the same initial interval. Where J_0^α denotes the Riemann-Liouville fractional integral operator of order $\alpha > 0$ centered at 0 as follows.

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If $g_n \to g$ uniformly on $[0,1]$ then $J^{\alpha} g_n \to J^{\alpha} g \;\therefore\; J_0^{\frac{1}{2}} g_n = J_0^{1/2} \sum_{i=0}^{n} \frac{x^{i+1/2}}{\Gamma(i+3/2)}$

$$= \sum_{i=0}^{n} J_0^{1/2} \sum_{i=0}^{n} \frac{x^{i+1/2}}{\Gamma(i+3/2)} = \sum_{i=0}^{n} \frac{\Gamma(i+3/2) x^{i+1}}{\Gamma(i+3/2)\Gamma(i+3/2+1/2)} = \sum_{i=0}^{n} \frac{x^{i+1}}{\Gamma(i+2)} = \sum_{i=0}^{n} \frac{x^{i+1}}{(i+1)!}$$

$$= \left(\sum_{i=0}^{n+1} \frac{x^i}{i!} \right) - 1 \text{ as } n \to \infty ,$$

$$J_0^{\frac{1}{2}} g_n(x) = e^x - 1 \qquad (3)$$

5.0 Fractional differential equations or Laplace transform

Considering the fractional differential equation of the type $D_0^{\alpha} y = -\lambda^{\alpha} y, \;\; y(0) = y_0$ and $0 < \alpha < 1$. First, we recognize that the Caputo fractional derivative can be written as a convolution equation

$$D_0^{\alpha} y = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{y'(x)}{(t-x)^{\alpha}} dx$$

$$D_0^{\alpha} y = \frac{1}{\Gamma(1-\alpha)} t^{-\alpha} * y'(t) = -\lambda^{\alpha} y . \qquad (4)$$

Using the notation $\ell(y) = \bar{y}$, we apply the Laplace transform to (4) and the result is obtained as follows.

$$\frac{1}{\Gamma(1-\alpha)} \ell\{t^{-\alpha} * y'(t)\} = -\lambda^{\alpha} \ell\{y\},$$

$$\frac{1}{\Gamma(1-\alpha)} \ell\{t^{-\alpha}\} \ell\{y'(t)\} = -\lambda^{\alpha} \bar{y}$$

$$\Rightarrow \frac{1}{S^{1-\alpha}} \left(S\bar{y} - y_0 \right) = -\lambda^{\alpha} \bar{y}$$

$\left(S^{\alpha} + \lambda^{\alpha}\right)\bar{y} = y_0 S^{\alpha-1} = \bar{y} = \dfrac{y_0 S^{\alpha-1}}{S^{\alpha} + \lambda^{\alpha}}$ Taking the inverse Laplace transform, it gives

$$y(t) = y_0 E_{\alpha}\left(-(\lambda t)^{\alpha}\right) \qquad (5)$$

5.1 Fractional integration

Suppose that $f(x) = (x-a)^{\beta}$, where $\beta > 0$ and let $n > 0$ then,

$$J_a^n f(x) = \frac{\Gamma(\beta+1)}{\Gamma(n+\beta+1)} (x-a)^{n+\beta} \qquad (6)$$

From the definition of the integral, we have $J_a^n f(x) = \dfrac{1}{\Gamma(n)} \displaystyle\int_a^x (t-a)^{\beta} (x-t)^{n-1} dt$

Let $t = a + s(x-a)$ then $dt = (x-a)ds$ and substituting that into the integral gives

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$$J_a^n f(x) = \frac{1}{\Gamma(n)} \int_0^1 [s(x+a)]^\beta \ (x-a)^{n-1}(1-s^{n-1})(x-a)ds \, .$$

$$J_a^n f(x) = \frac{(x-a)^{n+\beta}}{\Gamma(n)} \int_0^1 s^\beta (1-s)^{n-1} ds$$

But $\int_0^1 s^\beta (1-s)^{n-1} ds = \frac{\Gamma(n).\Gamma(\beta+1)}{\Gamma(n+\beta+1)}$ hence

$$J_a^n f(x) = \frac{\Gamma(\beta+1)}{\Gamma(n+\beta+1)} (x-a)^{n+\beta} \qquad (7)$$

5.2 Relationship between Beta function and gamma function

Here we shall use the Laplace transform to establish the relationship between the Beta function and the gamma function as presented below:

Suppose that $\beta(z,\omega) = \frac{\Gamma(z)\Gamma(\omega)}{\Gamma(z+\omega)}$ we can show that the RHS is equal to the LHS as follows:

Given $\beta(z,\omega) = \int_0^1 t^{z-1}(1-t)^{\omega-1} dt$ using convolution theory,

$\beta(z,\omega)(x) = \int_0^x t^{z-1}(1-t)^{\omega-1} dt$ Taking Laplace transform of both sides

$$\ell\{\beta(z,\omega)(x)\} = \ell\{t^{z-1}\}\ell\{t^{\omega-1}\} = \frac{\Gamma(z)}{s^z}\frac{\Gamma(\omega)}{s^\omega} = \frac{\Gamma(z)\Gamma(\omega)}{s^{z+\omega}}$$

Taking inverse Laplace transform of both sides

$$\ell^{-1}\{\beta(z,\omega)(x)\} = \ell^{-1}\left\{\frac{\Gamma(z)\Gamma(\omega)}{s^{z+\omega}}\right\}$$

$\beta(z,\omega) = \frac{\Gamma(z)\Gamma(\omega)}{\Gamma(z+\omega)} x^{z+\omega-1}$ Let $x = 1$

$$\beta(z,\omega) = \frac{\Gamma(z)\Gamma(\omega)}{\Gamma(z+\omega)} \qquad (8)$$

Hence, show that $\int_0^a \frac{dy}{\sqrt{a^4 - y^4}} = \frac{\Gamma(1/4)^2}{4a\sqrt{2\pi}}$

Suppose $\frac{y}{a} = t^{1/4}$, which gives

$$\int_0^a \frac{dy}{\sqrt{a^4 - y^4}} = \frac{1}{a^2}\int_0^a \left(1 - \frac{y^4}{a^4}\right)^{-\frac{1}{2}} dy$$

$$= \frac{1}{a^2}\int_0^1 (1-t)^{-\frac{1}{2}} \frac{at^{-3/4}}{4} dt = \frac{1}{4a}\int_0^1 (1-t)^{\frac{1}{2}-1} t^{\frac{1}{4}-1} t$$

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$$=\frac{1}{4a}\beta\left(\frac{1}{2},\frac{1}{4}\right)=\frac{1}{4a}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

$$\frac{1}{4a}\frac{\Gamma(1/2)\Gamma(1/4)^2}{\Gamma(3/4)\Gamma(1/4)}$$ Recollecting that $\Gamma(1/2)=\sqrt{\pi}$ and $\Gamma(3/4)\Gamma\left(\frac{1}{4}\right)=\frac{\pi}{\sin\frac{\pi}{4}}$

gives the result.

6.0 Conclusion

We conclude that the Riemann-Liouville derivative on $[a,b]$ is integrated before differentiating and the Caputo fractional derivative on $[a,b]$ is differentiated before integrating. It was also discovered that the Riemann-Liouville derivative will require the fractional integral of $\lim_{t\to a} J_a^{1-\alpha} f(t)$ as a necessary initial condition while the Caputo derivatives will only require $f(a)$ as an initial condition.

Finally, we observed that in calculating the fractional differential equation , the Caputo fractional derivative will first be written as a Convolution equation and both Laplace transform and inverse Laplace transform can be apply to obtain the general solution.

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